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Quantum Moduli Spaces of Linear and Ring Mooses*

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Abstract

Quantum moduli spaces of four dimensional $SU(2)^r$ linear and ring mooses with $\mathcal{N} = 1$ supersymmetry and link chiral superfields in the fundamental representation are produced starting from simple pure gauge theories of disconnected nodes.

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1 Introduction

Moose[1] diagrams give succinct graphical representations of the transformations of matter fields under gauge (and global) symmetries. The gauge symmetries are represented by nodes and the matter link fields by lines. The recent interest in mooses is because a class of moose diagrams has been shown to transform into a description of extra dimensions.[2, 3]. Furthermore, the "theory space" of mooses has been used to investigate various fundamental issues such as electroweak symmetry breaking and accelerated grand unification.[4]. The transformation of a moose diagram into a description of extra dimensions occurs when the link fields develop vacuum expectation value (vev) and "hop" across the nodes. It is well known that supersymmetric gauge theories have larger and richer moduli spaces of vacua[5] than non-supersymmetric gauge theories. Therefore, supersymmetric mooses could provide richer theory spaces for model building.

Our interest in this note is the construction of the moduli spaces of $\mathcal{N} = 1$ supersymmetric $SU(2)^r$ linear and ring mooses where the gauge group at each node is $SU(2)$ and the links are chiral superfields that transform as fundamentals under the nearest gauge groups and as singlets under the rest. We will obtain nontrivial quantum moduli space constraints by starting from pure gauge theories of disconnected nodes and exploiting simple and efficient integrating in[6, 7] and out procedures. Explicit parameterization of the vacua in terms of gauge invariant objects constructed out of the chiral superfields will be found. A generic point in the moduli space of the ring moose has an unbroken $U(1)$ gauge symmetry and the ring moose is in the Coulomb phase. We will find two singular submanifolds with modulus that is a nontrivial function of all the independent gauge invariant objects needed to parameterize the moduli space of the ring moose. The Seiberg-Witten elliptic curve that describes the quantum moduli space of the ring moose will follow from our computation.

Seiberg-Witten elliptic curves of the ring moose were computed in [8] using a different method where it was started with the curve for a ring with two nodes given in [9] and various asymptotic limits and symmetry arguments were used to generalize the curve to a ring with arbitrary number of nodes. Here we will directly and explicitly compute the singularities of the quantum moduli space and the corresponding Seiberg-Witten elliptic curve for a ring moose with arbitrary number of nodes. We believe that the curve in [8] is incorrect for ring mooses with four or more nodes.

An expanded version of this note that also includes more topics and results is presented in [10].

2 Quantum moduli space of the linear moose[1]

Consider an $\mathcal{N} = 1$ $SU(2)^r$ supersymmetric linear moose gauge theory with $SU(2)$ gauge group at each node and matter chiral superfield Q_i linking the i^{th} and $(i+1)^{th}$ nodes. The matter content of the linear moose is shown in Table 1. The equivalent moose diagram representation is shown in

[1] Quantum moduli space constraint relations for a linear moose with two and more nodes were first shown to us by Howard Georgi. Many results in this section overlap with results in [11].

Figure 1.

An internal link Q_i is a doublet that transforms as $(\square, \square)$ under $SU(2)_i \times SU(2)_{i+1}$ and as singlet under all the other gauge groups. The external superfields $Q_0 \sim \square$ under $SU(2)_1$ and $Q_r \sim \square$ under $SU(2)_r$. Each external link is two doublets with $SU(2)$ subflavor symmetry. We will compute the quantum moduli space of this theory starting from pure disconnected gauge groups and integrating in the matter fields.

	Gauge symmetries $SU(2)_1$	$SU(2)_2$	$SU(2)_3$	$\cdots$	$SU(2)_r$	Subflavor symmetries $SU(2)_1$	$SU(2)_2$
Q_0	$\square$	1	1	$\cdots$	1	$\square$	1
Q_1	$\square$	$\square$	1	$\cdots$	1	1	1
Q_2	1	$\square$	$\square$	$\cdots$	1	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
Q_r	1	1	1	$\cdots$	$\square$	1	$\square$

Table 1: Matter content of the linear moose.

Figure 1: Linear moose with r nodes and $r+1$ links. The external links Q_0 and Q_r each have one color and one subflavor indices and each internal link has two color indices.

Gaugino condensation in each pure gauge theory gives a nonperturbative superpotential,

$$W_{\rm d} = \sum_{i=1}^{r} 2\epsilon_i \Lambda_{0i}^3, \tag{2.1}$$

where $\epsilon_i = \pm 1$ label the two vacua due to the breaking of the Z_4 R symmetry to Z_2 and Λ_{0i} is the nonperturbative scale of $SU(2)_i$. Our notation for the dynamical scales is Λ_{0i} for the scale of $SU(2)_i$ with no link, Λ_{id} when there is one link, and Λ_i when there are two links attached. The scale Λ_i is related to Λ_{0i} by threshold matching of the gauge coupling running at the masses m_{i-1} and m_i of Q_{i-1} and Q_i respectively,

$$\Lambda_{0i}^6 = \Lambda_i^4 m_{i-1} m_i. \tag{2.2}$$

There are $r+1$ matter chiral superfields each with four complex degrees of freedom. We can construct a total of $\frac{1}{2}(r^2+3r+8)$ gauge singlets given by determinants of products of one to r consecutive link superfields, and the product of all the chiral superfields:

$$\det(Q_i), \tag{2.3}$$

$$\det(Q_iQ_{i+1}), \quad \cdots, \ \det(Q_0Q_1\cdots Q_{r-1}), \quad \det(Q_1Q_2\cdots Q_r), \tag{2.4}$$

$$\text{and} \quad Q_0Q_1\cdots Q_r. \tag{2.5}$$

For a generic linear moose, the gauge symmetry is completely broken and $3r$ of the complex degrees of freedom become massive or are eaten by the super Higgs mechanism. Consequently, there are only $4(r+1) - 3r = r+4$ massless complex degrees of freedom left. Because we have $\frac{1}{2}(r^2+3r+8)$ gauge singlets, there must be $\frac{1}{2}(r^2+3r+8) - (r+4) = r(r+1)/2$ constraints. We claim that a constraint involving the determinants of only subsegments of the moose chain given in (2.4) are not modified by the extra links and nodes. We can see that as follows: Consider the determinant of a subsegment $\det(Q_iQ_{i+1}\cdots Q_j)$. The color indices from the gauge groups $SU(2)_i$ and $SU(2)_j$ are not contracted with the colors of $SU(2)_{i-1}$ and $SU(2)_{j+1}$ respectively. Consequently, these adjoining gauge groups behave like global subflavor symmetries. This amounts to saying that as far as $\det(Q_iQ_{i+1}\cdots Q_j)$ is concerned, the moose chain is cut off at the $(i-1)^{\text{th}}$ and $(j+1)^{\text{th}}$ nodes. Therefore, finding a constraint for $\det(Q_iQ_{i+1}\cdots Q_j)$ is not an independent problem. Thus all the $r(r+1)/2$ moduli space constraints can be easily deduced from the one constraint which can be parameterized by the $r+5$ independent gauge singlets:

$$M_i \equiv \frac{1}{2}(Q_i)_{\alpha_i\alpha_{i+1}}(Q_i)_{\beta_i\beta_{i+1}}\epsilon^{\alpha_i\beta_i}\epsilon^{\alpha_{i+1}\beta_{i+1}} = \det(Q_i) \tag{2.6}$$

and

$$T_{fg} \equiv (Q_0)_{f\alpha_1}(Q_1)^{\alpha_1}_{\ \alpha_2}(Q_2)^{\alpha_2}_{\ \alpha_3}\cdots(Q_{r-1})^{\alpha_{r-1}}_{\ \alpha_r}(Q_r)^{\alpha_r}_{\ g}. \tag{2.7}$$

For M_0 and M_r one of the indices in Q_0 and Q_r is for subflavor.

In order to integrate in the link fields to the pure gauge theory of disconnected nodes, first we replace $\Lambda^3_{0i} \to (\Lambda^4_i m_{i-1}m_i)^{\frac{1}{2}}$ in (2.2). We then write

$$W = W_{\rm d}(\text{with}\,\Lambda^3_{0i} \to (\Lambda^4_i m_{i-1}m_i)^{\frac{1}{2}}) + W_{\text{tree,d}} - W_{\text{tree}}. \tag{2.8}$$

W_{tree} is a tree level superpotential that contains couplings to all the independent gauge invariants,

$$W_{\text{tree}} = \operatorname{tr}(c\,T_{(i,j)}) + \sum_{i=1}^{r+1} m_{i-1}M_{i-1}. \tag{2.9}$$

The term $W_{\text{tree,d}}$ is computed by choosing an arbitrary link chiral superfield Q_k and integrating out Q_k in the gauge and flavor invariant tree level superpotential[2] $\operatorname{tr}(c\,T) + m_kM_k$, where c is a constant 2×2 matrix. This gives

$$W_{\text{tree,d}} = -\frac{\det(c)}{m_k}\det(Q_0Q_1\ldots Q_{k-1})\det(Q_{k+1}Q_{k+2}\ldots Q_r). \tag{2.10}$$

We will see that the final result on the moduli space constraint does not depend on k. For simplicity of notation, we introduce a more general way of representing consecutive products of link chiral superfields and define

$$T_{(i,j)} \equiv Q_iQ_{i+1}\ldots Q_j \tag{2.11}$$

[2]This is discussed in detail in [10].

Note that $T_{(i,j)}$ is a 2×2 matrix with hidden indices. The superpotential we need for integrating in all the independent gauge singlets is then, putting (2.1), (2.9) and (2.10) in(2.8),

$$\begin{aligned} W &= 2\sum_{i=1}^{r}\epsilon_i(\Lambda_i^4 m_{i-1}m_i)^{\frac{1}{2}} - \frac{\det(c)}{m_k}\det T_{(0,k-1)}\det T_{(k+1,r)} \\ & -\mathrm{tr}\,(c\,T_{(i,j)}) - \sum_{i=1}^{r+1} m_{i-1}M_{i-1}. \end{aligned} \tag{2.12}$$

The integrating in procedure is completed by minimizing (2.12) with the coupling constants m_i and c.

Integrating out m_i and c in (2.12), recursively solving for m_i and c, and putting into (2.12) gives $W=0$ and a quantum moduli space constrained by

$$\det T_{(0,r)} - \frac{\det T_{(0,k-1)}\det T_{(k+1,r)}}{\Omega_{(0,k-1)}\Omega_{(k+1,r)}}\Omega_{(0,r)} = 0, \tag{2.13}$$

where we have introduced $\Omega_{(i,j)}$ functions to simplify our notation. The Ω functions are defined by

$$\begin{aligned} \Omega_{(i,j)} &\equiv \prod_{q=i}^{j} M_q - \sum_{p=i+1}^{j}\Big(\Lambda_p^4 \prod_{q\neq p-1,\,p} M_q\Big) + \sum_{p=i+1}^{j-2}\sum_{l=1}^{j}\Big(\Lambda_p^4\Lambda_{p+l+2}^4 \prod_{q\neq p-1,\,p,p+l+1,\,p+l+2} M_q\Big) \\ & - \cdots + (-1)^{(j-i+1)/2}\prod_{p=1}^{(j-i+1)/2}\Lambda_{i+2p-1}^4, \end{aligned} \tag{2.14}$$

if $j-i$ is odd, and

$$\begin{aligned} \Omega_{(i,j)} &\equiv \prod_{q=i}^{j} M_q - \sum_{p=i+1}^{j}\Big(\Lambda_p^4 \prod_{q\neq p-1,\,p} M_q\Big) + \sum_{p=i+1}^{j-2}\sum_{l=1}^{j}\Big(\Lambda_p^4\Lambda_{p+l+2}^4 \prod_{q\neq p-1,\,p,p+l+1,\,p+l+2} M_q\Big) \\ & - \cdots + (-1)^{(j-i)/2}\sum_{q=0}^{(j-i)/2}\Big(M_{i+2q}\prod_{p=0}^{2q-1}\Lambda_{i+2q-2p-1}^4 \prod_{l=1}^{(j-i)/2-q}\Lambda_{i+2q+2l}^4\Big), \end{aligned} \tag{2.15}$$

if $j-i$ is even. We take $j>i$ unless explicitly stated. When $i=j$, we have $\Omega_{(i,i)} = \det M_i$.

Thus the quantum moduli space is constrained by the recursion relations given by (2.13). Note that k in (2.13) is arbitrary and could take any value from 0 to r. As we have argued earlier in this section, a similar relation as (2.13) should hold for a subset of the linear chain, and we write a more general form of the moduli space constraints as

$$\det T_{(i,j)} - \frac{\det T_{(i,k-1)}\det T_{(k+1,j)}}{\Omega_{(i,k-1)}\Omega_{(k+1,j)}}\Omega_{(i,j)} = 0. \tag{2.16}$$

Now we can easily prove that the result (2.16) is independent of k, since we can repeatedly use the same recursion relations to simplify the fractional factor in the second term, and (2.16) gives

$$\det T_{(i,j)} - \Omega_{(i,j)} = 0. \tag{2.17}$$

Note that (2.17) gives $r(r+1)/2$ constraints that completely remove all the redundancy in the set of gauge singlets.

The first few Ω functions are

$$\Omega_{(i,\,i+1)} = M_i M_{i+1} - \Lambda_{i+1}^4 \tag{2.18}$$

$$\Omega_{(i,\,i+2)} = M_i M_{i+1} M_{i+2} - \Lambda_{i+1}^4 M_{i+2} - \Lambda_{i+2}^4 M_i \tag{2.19}$$

$$\begin{aligned}\Omega_{(i,\,i+3)} &= M_i M_{i+1} M_{i+2} M_{i+3} - \Lambda_{i+1}^4 M_{i+2} M_{i+3} - \Lambda_{i+2}^4 M_i M_{i+3} \\ &\quad -\Lambda_{i+3}^4 M_i M_{i+1} + \Lambda_{i+1}^4 \Lambda_{i+3}^4 \end{aligned}\tag{2.20}$$

$$\begin{aligned}\Omega_{(i,\,i+4)} &= M_i M_{i+1} M_{i+2} M_{i+3} M_{i+4} - \Lambda_{i+1}^4 M_{i+2} M_{i+3} M_{i+4} - \Lambda_{i+2}^4 M_i M_{i+3} M_{i+4} \\ &\quad -\Lambda_{i+3}^4 M_i M_{i+1} M_{i+4} - \Lambda_{i+4}^4 M_i M_{i+1} M_{i+2} \\ &\quad +\Lambda_{i+1}^4 \Lambda_{i+3}^4 M_{i+4} + \Lambda_{i+1}^4 \Lambda_{i+4}^4 M_{i+2} + \Lambda_{i+2}^4 \Lambda_{i+4}^4 M_i \end{aligned}\tag{2.21}$$

3 Quantum moduli space of the ring moose

Now we can construct the quantum moduli space of the ring moose starting from the linear moose. The matter content of the ring moose is shown in Table 3. The equivalent moose diagram is shown

	Gauge symmetries					
	$SU(2)_1$	$SU(2)_2$	$SU(2)_3$	$\cdots$	$SU(2)_{r-1}$	$SU(2)_r$
Q_0	$\square$	1	1	$\cdots$	1	$\square$
Q_1	$\square$	$\square$	1	$\cdots$	1	1
Q_2	1	$\square$	$\square$	$\cdots$	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
Q_{r-1}	1	1	1	$\cdots$	$\square$	$\square$

Table 2: Matter content of the ring moose.

in Figure 2. There are $r+1$ independent gauge singlets given by M_i defined in (2.6) and

$$U_{(0,\,r-1)} \equiv \frac{1}{2}(Q_0)_{\alpha_0\beta_0}(Q_1)_{\alpha_1\beta_1}(Q_2)_{\alpha_2\beta_2}\cdots(Q_{r-1})_{\alpha_{r-1}\beta_{r-1}}\epsilon^{\beta_0\alpha_1}\epsilon^{\beta_1\alpha_2}\cdots\epsilon^{\beta_{r-1}\alpha_0}. \tag{3.1}$$

We will start with the quantum moduli space constraint of the linear moose we found in Section 2. We will then integrate out the external links. Finally, a link field that transforms as $(\square, \square)$ under $SU(2)_r \times SU(2)_1$ will be integrated in to build the ring moose shown in Figure 2. Since we can at the same time obtain the superpotential for a linear moose with only one external link, let

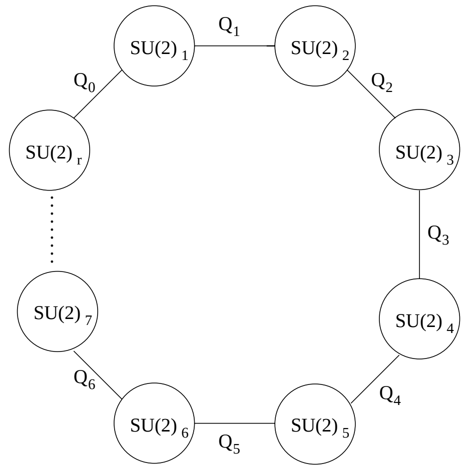

Figure 2: Ring moose with r nodes. Each link has two color indices.

us first integrate out only Q_r. The superpotential for the linear moose without Q_r is obtained by integrating out M_r, $T_{(0,\,r)}$ and A in

$$W = A\Big(\det T_{(0,\,r)} - \Omega_{(0,\,r)}\Big) + m_r M_r. \tag{3.2}$$

The resulting superpotential is

$$W = \frac{\Lambda_{rd}^5 \Omega_{(0,\,r-2)}}{\Omega_{(0,\,r-1)}}, \tag{3.3}$$

where $\Lambda_{rd}^5 = \Lambda_r^4 m_r$. Next we integrate out Q_0 by adding $m_0 M_0$ to (3.3) and minimizing with M_0 which gives the superpotential of the linear moose without the external links,

$$W = \frac{\Lambda_{1d}^5 \Omega_{(2,\,r-1)}}{\Omega_{(1,\,r-1)}} + \frac{\Lambda_{rd}^5 \Omega_{(1,\,r-2)}}{\Omega_{(1,\,r-1)}} \pm 2\frac{(\Lambda_{1d}^5 \Lambda_{rd}^5 \prod_{i=2}^{r-1} \Lambda_i^4)^{1/2}}{\Omega_{(1,\,r-1)}}. \tag{3.4}$$

The superpotential (3.4) can be interpreted as follows: For the moose chain with only internal links, the original $SU(2)^r$ gauge symmetry is completely broken and there is a new unbroken diagonal $SU(2)_D$. The first term comes from a single instanton in the broken $SU(2)_1$ and infinite series of multi-instantons from the broken $SU(2)_2$ to $SU(2)_{r-2}$. Similarly, the second term comes from a single instanton in the broken $SU(2)_r$ and an infinite series of multi-instantons from the broken $SU(2)_2$ to $SU(2)_{r-2}$. These can be seen by using the explicit form of the Ω functions and making an expansion of $\Omega_{(1,\,r-1)}^{-1}$ in powers of the scales of $SU(2)_2$ to $SU(2)_{r-2}$. The last term comes from gaugino condensation in the unbroken diagonal $SU(2)_D$. In fact, we can read off from (3.4) that the scale of the diagonal $SU(2)_D$ is

$$\Lambda_D = \Big(\frac{(\Lambda_{1d}^5 \Lambda_{rd}^5 \prod_{i=2}^{r-1} \Lambda_i^4)^{1/2}}{\Omega_{(1,\,r-1)}}\Big)^{\frac{1}{3}}. \tag{3.5}$$

Finally, we can construct the quantum moduli space of the ring moose shown in Figure 2 by integrating in Q_o. The new gauge singlets that appear in the ring moose which were not in the linear moose with only internal links are M_0 and $U_{(0,r-1)}$ and the tree level superpotential we need is

$$W_{\rm tree} = b\,U_{(0,r-1)} + m_0 M_0, \tag{3.6}$$

where b and m_0 are constants. The $W_{\rm tree,d}$ we need, because of the non-quadratic gauge singlet $U_{(0,r-1)}$, is obtained by minimizing $b\,U_{(0,r-1)} + m_0 M_0$ with Q_0 which gives $W_{\rm tree,d} = -\frac{b^2}{4m_0}\Omega_{(1,r-1)}$. The quantum moduli space constraint of the ring moose is then obtained by minimizing

$$\begin{aligned} W &= \frac{m_0\Lambda_1^4\Omega_{(2,r-1)}}{\Omega_{(1,r-1)}} + \frac{m_0\Lambda_r^4\Omega_{(1,r-2)}}{\Omega_{(1,r-1)}} \pm 2\frac{m_0(\prod_{i=1}^r\Lambda_i^4)^{1/2}}{\Omega_{(1,r-1)}} \\ &\quad -\frac{b^2}{4m_0}\Omega_{(1,r-1)} - m_0 M_0 - b\,U_{(0,r-1)} \end{aligned} \tag{3.7}$$

with m_0 and b which gives $W = 0$ and

$$U_{(0,r-1)}^2 + \Lambda_1^4\Omega_{(2,r-1)} + \Lambda_r^4\Omega_{(1,r-2)} - M_0\Omega_{(1,r-1)} \pm 2(\prod_{i=1}^r\Lambda_i^4)^{1/2} = 0. \tag{3.8}$$

This is symmetric in all links and scales.

Before we interpret (3.8), let us first recall the Seiberg-Witten hypothesis on the elliptic curve of an $SU(2)$ gauge theory. According to the Seiberg-Witten hypothesis[12], the quantum moduli space of an $SU(2)$ gauge theory coincides with the moduli space of the elliptic curve $y^2 = (x^2-u)^2 - \Lambda^4$. The singularities of this curve are given by the zeros of the discriminant $\Delta_\Lambda = (u^2-\Lambda^4)(2\Lambda)^8$. This occurs at $u = \pm\Lambda^2$ and $u = \infty$. The first two singularities at $u = \pm\Lambda^2$ are in the strong coupling region, and there is a massless monopole at one and a massless dyon at the other of these singularities. The singularity at $u = \infty$ is in the semi-classical region.

Now let us rewrite (3.8) as

$$u_r = \pm\Lambda_{(1,r)}^2, \tag{3.9}$$

where

$$u_r = U_{(0,r-1)}^2 + \Lambda_1^4\Omega_{(2,r-1)} + \Lambda_r^4\Omega_{(1,r-2)} - M_0\Omega_{(1,r-1)} \tag{3.10}$$

and

$$\Lambda_{(1,r)}^2 = 2(\prod_{i=1}^r\Lambda_i^4)^{1/2} \tag{3.11}$$

Note that the modulus u_r contains all the independent gauge invariants we needed to parameterize the moduli space of the ring. What (3.9) is telling us is that the function u_r is locked at $\pm\Lambda_{(1,r)}^2$. In other words, (3.9) gives two r - complex dimensional singular submanifolds in the $r+1$ - complex dimensional moduli space spanned by all the independent gauge invariants. A generic point in the moduli space of the ring moose has unbroken $U(1)$ gauge symmetry and the ring moose is in the Coulomb phase. Furthermore, giving large vevs to the link fields breaks the original $SU(2)^r$ gauge symmetry into a diagonal $SU(2)_D$ with matter in the adjoint representation. The two singularities given by (3.9) on the u_r plane can be nothing but the two singularities in the strong coupling region of the $SU(2)_D$ gauge theory with $\mathcal{N}=2$ supersymmetry. The monodromies around these singularities on the u_r plane must be the same as in Seiberg-Witten and the charge at the singularity $u_r = +\Lambda^2_{(1,r)}$ is that of a monopole and the charge at $u_r = -\Lambda^2_{(1,r)}$ coincides with a dyon. Having obtained these singularities and because the $U(1)$ coupling coefficient is holomorphic, we have determined the elliptic curve that parameterizes the Coulomb phase of the ring moose. Thus the quantum moduli space of the ring moose can be parameterized by the elliptic curve

$$y^2 = \left(x^2 - [U^2_{(0,r-1)} + \Lambda_1^4\Omega_{(2,r-1)} + \Lambda_r^4\Omega_{(1,r-2)} - M_0\Omega_{(1,r-1)}]\right)^2 - 4\prod_{i=1}^{r}\Lambda_i^4. \tag{3.12}$$

Note that although the singularities look the same as in Seiberg-Witten on the u_r plane, they are r - complex dimensional submanifolds with very non-trivial modulus given by (3.10). Using the definition (3.10) for u_r and the Ω functions given in Section 2, the first few u functions are

$$\begin{aligned}
u_2 &= U^2_{(0,1)} + \Lambda_1^4 + \Lambda_2^4 - M_0M_1, && (3.13)\\
u_3 &= U^2_{(0,2)} + \Lambda_1^4M_2 + \Lambda_2^4M_0 + \Lambda_3^4M_1 - M_0M_1M_2, && (3.14)\\
u_4 &= U^2_{(0,3)} + \Lambda_1^4M_2M_3 + \Lambda_2^4M_0M_3 + \Lambda_3^4M_0M_1 + \Lambda_4^4M_0M_2 \\
&\quad -\Lambda_1^4\Lambda_3^4 - \Lambda_2^4\Lambda_4^4 - M_0M_1M_2M_3, && (3.15)\\
u_5 &= U^2_{(0,4)} + \Lambda_1^4M_2M_3M_4 + \Lambda_2^4M_0M_3M_4 + \Lambda_3^4M_0M_1M_4 + \Lambda_4^4M_0M_1M_2 \\
&\quad +\Lambda_5^4M_1M_2M_3 - \Lambda_1^4\Lambda_3^4M_4 - \Lambda_1^4\Lambda_4^4M_2 - \Lambda_2^4\Lambda_5^4M_3 - \Lambda_3^4\Lambda_5^4M_1 \\
&\quad -\Lambda_2^4\Lambda_4^4M_0 - M_0M_1M_2M_3M_4. && (3.16)
\end{aligned}$$

Curves for the the ring moose were computed in [8] using a different method. A method used in [9] to obtain the curve for the $r = 2$ ring was continued in [8] for $r = 3$ and a generalization to arbitrary r was made. The idea was as follows: Because giving large vevs to the link fields breaks the $SU(2)^r$ gauge symmetry into a diagonal $SU(2)_D$ with matter in the adjoint representation, the theory in effect becomes that of a single $SU(2)$ with $\mathcal{N} = 2$ supersymmetry. The curves were then obtained by taking various asymptotic limits of the gauge singlet fields and the nonperturbative scales, comparing with the $\mathcal{N} = 2$ $SU(2)$ curve and imposing symmetries. Our results agree with [9] for $r = 2$ and with [8] for $r = 3$. However, we do not agree with [8] for $r \geq 4$. Only the first few terms in u_r were obtained in [8], which would give incorrect singular submanifolds in moduli space. Here we have obtained the quantum moduli space directly by integrating in all the independent link fields starting from a pure gauge theory of disconnected nodes and building the ring moose via the linear moose. This is done for a ring with arbitrary number of nodes without any need of imposing symmetries in the nodes or links and without taking asymptotic limits. Our result is exact in the strong coupling region and automatically symmetric in all nodes and links.

4 Summary

We have produced nontrivial quantum moduli spaces for $\mathcal{N} = 1$ supersymmetric $SU(2)^r$ linear and ring mooses starting from simple pure gauge theories of disconnected nodes by integrating in all matter link fields. For the ring moose, we obtained two singular submanifolds with a modulus that is a function of all the independent gauge singlets we needed to parameterize the quantum moduli space. The Seiberg-Witten elliptic curve that describes the quantum moduli space of the ring moose followed from our computation. More details and results are presented in [10].

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